Note on Invariants of the Weyl Tensor

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Abstract

Algebraically special gravitational fields are described using algebraic and differential invariants of the Weyl tensor. A type III invariant is also given and calculated for Robinson-Trautman spaces.

Key words : invariants, algebraic classification of the Weyl tensor.

1 Introduction

It is well known (see [1] and [2]) that the two algebraic invariants

$$I = \frac{1}{4}{}^{+}C_{abcd}{}^{+}C^{abcd}, \quad (1)$$

$$J = \frac{1}{8}{}^{+}C_{abcd}{}^{+}C^{cd}{}_{ef}{}^{+}C^{efab}, \quad (2)$$

where ${}^{+}C$ is the self-dual part of the Weyl tensor, provide a partial classification of the Weyl tensor. Moreover, if χ is the cross ratio of any four null directions then

$$I^3\left[(\chi+1)(\chi-2)\left(\chi-\frac{1}{2}\right)\right]^2 = 6J^2\left[(\chi+\omega)\left(\chi+\omega^2\right)\right]^3 \quad (3)$$

where $\omega = e^{\frac{2\pi i}{3}}$. In particular $I^3 = 6J^2 \neq 0 \Leftrightarrow \chi \in \{0, 1, \infty\} \Longleftrightarrow$ (2,1,1) or (2,2).

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2 Classification

For any F_{abcd} with symmetries similar to the ones of ${}^{+}C$ we define

$$J_F = F_{abcd;rs} F^{abcd}{}_{;tu} \overline{F}_{efgh;}{}^{rs} \overline{F}^{efgh;tu}; \tag{4}$$

remark that for any null field F, J_F is the invariant J in [3]. We are particularly interested in J_A, J_B and $J_{{}^{+}C}$ where

$$\begin{aligned} A_{abcd} &= I B_{abcd} - J\,{}^{+}C_{abcd} && (5)\\ B^{ab}{}_{cd} &= \frac{1}{2}{}^{+}C^{ab}{}_{rs}{}^{+}C^{sr}{}_{cd} - \frac{1}{3} I\,{}^{+}\delta^{ab}_{cd} && (6) \end{aligned}$$

where $\delta^{ab}_{cd} = \frac{1}{2}\left(g_{ad}g_{bc} - g_{ac}g_{bd} - i\eta_{abcd}\right)$, η being the Levi-Civita tensor. Notice that when $I^3 = 6J^2$ the tensor A is null ($A_{abcd} = 6\Psi_2^2(3\Psi_2\Psi_4 - \Psi_3^2)N_{ab}N_{cd}$) in the case (2,1,1) and it vanishes in the more degenerate cases (see [1]); for $I = J = 0$ the tensor B is null ($B_{abcd} = -4\Psi_3^2 N_{ab}N_{cd}$) in the (3,1) case and zero otherwise. Moreover

$$\begin{aligned} J_A &= \left|96\Psi_2^2(3\Psi_2\Psi_4 - \Psi_3^2)\rho^2\right|^4, && (7)\\ J_B &= \left|8\Psi_3\rho\right|^8. && (8) \end{aligned}$$

In conclusion, for space-times admitting an expanding congruence we have the following classification:
- $I^3 \neq 6J^2$, $I \neq 0$, $J \neq 0$: (1,1,1,1);
- $I^3 = 6J^2 \neq 0$, $J_A \neq 0$: (2,1,1);
- $I^3 = 6J^2 \neq 0$, $J_A = 0$: (2,2);
- $I = J = 0$, $J_B \neq 0$: (3,1);
- $I = J = 0$, $J_B = 0$, $J_{{}^{+}C} \neq 0$: (4);
- $I = J = 0$, $J_B = 0$, $J_{{}^{+}C} = 0$: (-).

3 Further remarks on the (3,1) case

For $I = J = 0$ case we can alternatively use the first order invariant obtained in [4]

$$J_P = C^{abcd;e} C_{amcn;e} C^{lmrn;s} C_{lbrd;s} \tag{9}$$

to distinct (3,1) case from more degenerate ones.

We did not investigate systematically invariants of second order but we mention that if

$$D_{rst} = {}^{+}C_{abcd;r}{}^{+}C^{abcd}{}_{;st} \tag{10}$$

then

$$D = D_{[rs]t} \overline{D}^{[rs]t} \tag{11}$$

has the following expression for a (3,1) Robinson-Trautman solution with $P = P(\sigma, \xi, \eta)$:

$$
\begin{aligned}
D &= \frac{36p^4}{r^{14}} \left(K_\xi^2 + K_\eta^2\right) \left[\frac{1}{8} \left(K_\xi^2 + K_\eta^2\right) K + p \left(K_{\xi\eta}^2 - K_{\xi\xi} K_{\eta\eta}\right)\right] \qquad (12) \\
&\quad + 9 \frac{p^4}{r^{13}} \left[\left(K_\xi^2 + K_\eta^2\right)^2\right]_{,\sigma} .
\end{aligned}
$$

Remark that for (3,1) and (4) cases, the geometry of each light cone is independent of the one of its neighbors; and both J_P and J_B depend only on the geometry of each individual light cone. However, the invariant D also depends on the rate of change of the geometry from one light cone to another.

References

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